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CONTINUITY AND NUMBER.

THE problem of continuity enters into mathematics, physics, and philosophy in one guise or another. It seems to have reached a satisfactory solution only in the first-named field, where the theory of continua of higher order, and the Dedekind theory of the nature of irrationals, appear to have brought about that much-desired freedom from paradoxes which the physicist and philosophical geometer cannot attain. These latter investigators are, therefore, justified in investigating the mathematician's solution of the problem in the hope of gleaning some fruitful suggestion or analogy. It is in this spirit that the present study is undertaken, the aim being not so much to produce a mathematical criticism as to find what the bearings may be which the mathematicians' discoveries have upon the philosophical issues centering about continuity. The conclusion we shall reach is strictly a negative one, when viewed from the philosophical standpoint: The concept of continuity appearing in all discussions in the theory of numbers is totally unlike the concept similarly named which appears in geometry and theoretical physics ; it is, indeed, not continuity at all that is being spoken of. Accordingly, all higher forms of speculation which make use of the arithmetical concept (higher mathematics and metaphysics of the idealistic school, for instance) must be subjected to revision in the light of our amendments.

In criticising the theory of number we shall — perhaps with some injustice to mathematicians as a class — investigate only the well-known and widely accepted results of Dedekind's studies on the nature of number, which embody the most typical views of modern theorists, generally speaking.[1] The starting-point is the usual one, a statement of the symbolic nature of number. "The number concept," says Dedekind, is "entirely independent of the notions or intuitions of space and time, . . . an immediate result from the laws of thought, . . . [numbers are] free creations of the human mind; they serve as a means of apprehending more easily and more sharply the difference of things."[1] To the psychological reader of these words, the distinction between number as an immediately felt *quale*, and number as a sign of this *quale*, is not clearly drawn; it is hardly exact, moreover, to pronounce number, the quantity *quale*, entirely independent of the notions of space and time. With respect to this last point, however, we may well grant the correct intention of the mathematician, which is, it seems, to say that quantity, as such, is neither essentially spatial nor temporal. The concept of number, as expression of pure quantity, is such that the number symbols may be applied to anything which can be discriminated. On this point mathematician and psychologist may readily agree. For the present, we waive a discussion of some other peculiarities of pure quantity which are involved in the current mathematical concept; we may say that quantity is a phase of all thinkable things in some sense or other.

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[1] It is only fair to state that some mathematicians, for instance Hilbert, are not content with the Dedekind theories; yet, so far as I can understand their points of difference, none of them are free from the confusion of two or more concepts of continuity.

There is a second preliminary point which is not emphasized by the theorists, however, but which is vital to a logical discussion of arithmetical continuity; I refer to the commonplace fact that, if operations are to be performed with numbers in the number-system, whatever the base (= 1) is taken to represent in the particular series of reasonings, that same represented thing must be the sole true element in all groups symbolized by all the other members of the system. The schoolma'am expresses this by saying you must not add two apples to four chairs. I say that the higher theories of number have not given due weight to this fundamental rule of logic; in order to make good this assertion, we need but turn to our theme directly and see the curious results of the neglect in the discussions of numerical continuity.

Turning first to the whole-number series, we find the mathematician saying that this is discrete, full of gaps — in short, a poor apology for a continuum. In order to make it truly continuous, elaborate interpolations must be made; fractions and irrationals must be inserted 'between' integers;[1] whereupon we have a true number continuum of which it may be said that "the four fundamental [arithmetical] operations are always performable with any two individuals in R [the system], *i. e.*, the result is always an individual of R"; and "if a, c are two different numbers, there are infinitely many different numbers lying between a, c."[2] Now, from the way in which the perfect number continuum is built up and from what is said about it, I think it can be shown that mathematicians have not remained true to their first assumptions about the nature of number as a symbol. If it can be shown that interpolation between integers involves a primary logical difficulty, then the notion of arithmetical continuity must undergo revision. I think it can be demonstrated that there is involved the fallacy of equivocation, in that numbers are employed in a single chain of reasoning, with various symbolic values. In order to make this plain, it will prove expedient to explain in what sense a pair of numbers is said to be adjacent or proximate in a continuum.

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[1] *Essays on the Theory of Numbers*, Engl. transl., p. 31.

Let us accept the mathematician's definition of continuity in number: "If the system R of all real numbers breaks up into two classes A_1, A_2 such that every number a_1 of class A_1 is less than every number a_2 of class A_2, then there exists one and only one number a by which this separation is produced."[3] This is stated about all real numbers as a system, *i. e.*, about the real-number system. But it is equally applicable to the system of whole numbers, provided we bear in mind the injunction that the base must have a constant reference in each particular chain of reasonings. For then, if the *number* 1 represents the group θ composed of the single element θ, and if every number in the series represents a group whose elements are θ and hence homogeneous, it follows that there is no group whose elements are each θ whose group-characteristic, *however this may be determined*, is in any sense of the word 'intermediate' between the characteristic of the group of $n\theta$ and that of the group $(n + 1)\theta$. This means that, with a given meaning (reference) for the base (1), every whole number is the unique separator of two classes which comprise the whole whole-number system. We may even go farther and say that, in a certain sense, every whole number is the limit which those two classes approach; but this gives a meaning to the term 'limit,' which is somewhat at variance with its usual mathematical import, so that it is not advisable to urge it. All we wish to insist upon is that, whenever the base is made a true symbol (of anything whatsoever), then n and $n+1$ are proximates in the system. For if all the groups represented by the numbers of the system are by definition composed of homogeneous elements θ, then no group not thus composed can find a representative number in that system.

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[1] For a very brief popular presentation of the development of a higher continuum, *cf.* Poincaré, *Science and Hypothesis*, Ch. II.

[2] Dedekind, *Op. cit.*, pp. 5 f.

[3] *Ibid.*, p. 20.

Whatever the base symbolizes, therefore, fractional numbers cannot consistently be interpolated in the whole-number series. For, given any base-value whatsoever, such fractions are absolutely meaningless. Hence we may generalize by declaring that interpolation of 'values' between integers is an illogical procedure. It now remains for us to show what the actual and undeniable *meaning* of fractions (and irrationals) is. It will require considerable politeness on the part of the mathematician to listen patiently to the remarks I have to make in elucidation of these points; for it is no trivial thing for him to reject the orthodox proposition that "if α, γ are any two different numbers, then there exist infinitely many different numbers β lying between α, γ."[1] Nevertheless, I think the kernel of truth in this misstatement may be separated from the chaff in such a way that the only kind of 'continuity' and 'infinity' needed by the mathematician will survive the winnowing.

Looking at the nature of the number-base (1) and its symbolic function, we discover that 1 may be taken to represent absolutely anything that can be, in any guise, an object of attention or reference. We must thus speak quite logically and mathematically of 'seven unthinkable things,' for all that is necessary for the mathematician is that he be able to *refer* to them.[2]

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[1] *Op. cit.*, p. 19.

[2] We need not be disturbed by the following paradox: It is imposible to number unthinkable things because some unthinkable things are not amenable to the laws of

As every different element θ may serve as real base to the symbol 1, we may say that we have as many possible systems as there are possible elements. By 'system' we mean a genus whose species differ solely with respect to the number of identical individuals (elements θ) which each of these species contains. We are thus barring by definition such things as mixed systems, *e. g.*, those the elements of whose species are heterogeneous or those whose species are determined by some other principle than that of the mere number of identical elements.

With reference to any arbitrarily chosen element θ, we may then say that it can be the measure of all possible systems, because it may be regarded as the 1-group, the 2-group, or the n-group of a system; and, on the other hand, when regarded as the 1-group, each successive 'transform' — *e. g.*, the 2-group, 3-group, etc., having this element as base — may be regarded as a new base for another system. All such systems may then be named with reference to the original arbitrarily chosen basic element; *e. g.*, when θ is base, the system is $N(\theta)$; when θ is the 2-group, the system is $N(\frac{\theta}{2})$; when θ is the n-group, the system is $N(\frac{\theta}{n})$; and again, taking transforms of θ as bases, we have such systems as $N(2\theta)$, $N(n\theta)$, etc. Finally, inasmuch as absolutely anything may be chosen as arbitrary first base, it is clear that every possible pure system is capable of description in such a scheme; but in order to construct a system which shall include absolutely everything numerable, we must take as base for our universal measure simply 'an object of reference,' *i. e.*, 'a thing referred to (by the particular thinker).'

A peculiarity of such a system must now be mentioned. Calling each individual member of the system a 'group' — *e. g.*,

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arithmetic. *E. g.*, a thing which, when added to itself once, gives thrice itself as result is an unthinkable thing. The student of logic will perhaps solve this without difficulty. It is then clear that anything which is part of some higher system may equally well be referred to and symbolized by 1. Hence whatever elements of whatever things there are, each one may be the object referred to by unity. And we need not construe elements in the narrow way a physicist or chemist might; we can signify thereby any 'aspect,' 'phase,' 'quality,' or 'fragment' of absolutely anything. Only, in doing this we must be very careful to retain the same base-value throughout every step in our reasonings which have reference to the same subject matter.

1 = the 1-group, n = the n-group — we see that every group may be modified in two different ways:

1. By the addition or removal of n of its proper elements (*i. e.*, of those elements which are identical with the base of the system to which the group belongs), in which case it is transformed into another group of the same system.

2. By the addition or removal of n of the sub-elements of some of its proper elements, whereby the resultant aggregate is no longer a true member of the original system. Assuming as we do in this case that merely some part of an original element is added or removed, we see at once that the resultant aggregate cannot be described in terms of the definitions of the original system, but demands some other basic element. The smallest sub-element that was added or removed must now be taken as the base for that system to which the above-named resultant aggregate properly belongs. The new base must logically be symbolized by unity; but, so long as we are reasoning at once about two systems, the old and the new bases must be distinguished, *e. g.*, by writing them 1 and $1'$ respectively. The unity appearing in a fractional expression, it will now be seen, is not the same as the unity appearing in an integral one; the resemblance is a relative one only. Each unity is the base of a different system, and these systems are related in a way wholly accidental to arithmetic.

The logical meaning of incommensurability becomes apparent now, and we refer to it because it probably makes clear the meaning of fractions generally and the bad logic of talking about interpolation of fractional values. To say that 3 and 10 of any given system are incommensurable means that, if the 3-group of that system be taken as base of a new system, then the 10-group does not appear in this new system. This suggests at once that there are two degrees or types of incommensurability, one relative and the other absolute; relative incommensurability is that found between groups neither of which can be converted into a base of any system in which the other appears as a true member; absolute incommensurability, on the other hand, is that found between groups which cannot both be true members of any system whatsoever, be the base of such system what it may. Thus, to say that 1 and $\sqrt{2}$ are incommensurable means that, whatever the the base may be (symbolized by 1), no $\sqrt{2}$-group appears in the system built up on this base. This is absolute incommensurability, because it lacks a characteristic of relative incommensurability, namely, commensurability in terms of some base other than the particular groups (numbers) termed incommensurable. The process of finding the least common denominator expresses the process of transforming incommensurables of one system into commensurables of another.

Bearing this in mind, we may see from a new standpoint the utter impropriety of constructing a number continuum by interpolation between integers. A fraction is, as Ehrenfels has well suggested,[1] a 'qualified number,' to be distinguished from pure number in that the former contains information about the qualities of the represented thing. That is to say, ½ means 'the 1-group in that system whose base is one of the two like elements in the base of another system previously fixed upon, known, and referred to.' In short, paradoxically enough, ½ means 1, but 1 of another system; it means 1′, we might more accurately say. Its value is dependent upon that of pure unity, whatever that may chance to be, and it is meaningless save in reference to pure unity. Is it not a logical necessity to conclude from this that fractions do not belong to the same system as integers do, and that the attempt to unite fractions and integers into one systematic continuum at least involves some peculiar, extra-arithmetical assumptions about the material *represented?* This same suspicion has occurred to Poincaré, who has said: "Should we have the notion of fractional numbers if we had not previously known a *matter* that we conceive as infinitely divisible, that is to say, a continuum?"[2] In other words, the arithmetical systems of integers and fractions are suspected of belonging together only by virtue of the nature of the thing symbolized by them. But, curiously enough, it has not been seen that even that most continuous and dissectible object called 'pure quantity' cannot justify the construction of the conventional number continuum, inasmuch as the objects of reference of fractions with different denominators are different, so that in strict logic there is no conceivable sense in which $1/n$ and $1/(n+1)$ belong to the same system or continuum. Only when we think of the reference-object itself as capable of continuous variation in quantity, do we think of each possible increment as homogeneous or as 'belonging to' that same object; but what has this continuity and homogeneity to do with the number continuum? Plainly nothing more than this: The number system, being symbolic of quantities, represents numerically all quantitative differences in the represented matter; but this infinite elaboration of quantitative differences can be logically represented only by means of a single base which shall symbolize 'the minimum increment,' more usually termed an infinitesimal, in which case we find the pure quantity continuum symbolized by the system of integers alone. Fractions, in other words, do not serve to symbolize pure quantity in the least, but express relations between different systems of (possibly) pure quantity. Their function being, therefore, different from that of pure number, which symbolizes in the limiting case pure quantity, fractional numbers do not belong logically to the same system as do integers.

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[1] *Vierteljahrss. für wiss. Phil.*, Bd. XV, p. 308.

[2] *Science and Hypothesis*, Ch. II.

We now have to ask the question, in conclusion, which is of prime philosophical interest: What does continuity mean in pure arithmetic? And how does it relate to the continuity spoken of, say, in geometry or physics? Our above observations lead to the view that the integral system forms as true a continuum as possible in the realm of number. Letting n = an integer, we can say that, with any accepted object of reference symbolized by the base ($= 1$) of the system to which n belongs, there can be no number referring ultimately to that same kind of object which is not an integer. Cast into logical language, this means simply that, within any given universe of discourse, the same term must refer to the same object. To say, in arithmetic, that there are infinite numbers between 1 and 2, is to use the word 'number' in a flagrantly ambiguous sense, now for an expression of pure quantity, and now for a sign of relation between different systems of quantities. On the other hand, to say logically that between 1 and 2 there are no numbers is to use the word 'number' univocally; 1 and 2 are continuous or 'adjacent,' if you will in the purely logical sense that there are no numbers of the same kind (pure numbers) which are greater than 1 and less than 2. In still more logical phraseology, we might express this by saying that, in any universe of discourse, it is impossible to have more than one object of reference and less than two. The meaning of this remark, though abbreviated, is probably plain enough.

Our conclusion, then, is that arithmetical continuity, even when correctly stated in its simplest logical form, throws no light upon the meaning of geometrical and physical continuities. And no description of a real continuum in terms of one-to-one correspondence with the real number system *explains* the character of continuity; such a description is only a formal *confession* of the presence of continuity. And the reason for this is that arithmetical continuity is really nothing but a peculiar instance of logical consistency of reference. It seems clear enough that spatial and temporal continuities are incapable of any such reduction to a merely logical demand. For all this, strictness in use of reference terms may well be advantageous in discussing these genuine continuities. Numerical description may be correct so far as it goes, but at best it gives us an aspect of continua which is not that of their continuity. We do not have to regard the numerical aspect as 'merely sujective'; it is simply other than the aspect of real continuity.

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